

Koszul duality in deformation quantization, I

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Abstract

Let α be a polynomial Poisson bivector on a finite-dimensional vector space V over \mathbb{C} . Then Kontsevich [K97] gives a formula for a quantization $f \star g$ of the algebra $S(V)^*$. We give a construction of an algebra with the PBW property defined from α by generators and relations. Namely, we define an algebra as the quotient of the free tensor algebra $T(V^*)$ by relations $x_i \otimes x_j - x_j \otimes x_i = R_{ij}(\hbar)$ where $R_{ij}(\hbar) \in T(V^*) \otimes \hbar\mathbb{C}[[\hbar]]$, $R_{ij} = \hbar \text{Sym}(\alpha_{ij}) + \mathcal{O}(\hbar^2)$, with one relation for each pair of $i, j = 1 \dots \dim V$. We prove that the constructed algebra obeys the PBW property, and this is a generalization of the Poincaré-Birkhoff-Witt theorem. In the case of a linear Poisson structure we get the PBW theorem itself, and for a quadratic Poisson structure we get an object closely related to a quantum R -matrix on V . At the same time we get a free resolution of the deformed algebra (for an arbitrary α).

The construction of this PBW algebra is rather simple, as well as the proof of the PBW property. The major efforts should be undertaken to prove the conjecture that in this way we get an algebra isomorphic to the Kontsevich star-algebra.

1 The main construction

1.1

First of all, recall here the Stasheff's definition of the Hochschild cohomological complex of an associative algebra A .

Consider the shifted vector space $W = A[-1]$, and the cofree coassociative coalgebra $C(W)$ (co)generated by W . As a graded vector space, $C(W) = T(A[1])$, the free tensor space. The coproduct is:

$$\Delta(a_1 \otimes a_2 \otimes \cdots \otimes a_k) = \sum_{i=1}^{k-1} (a_1 \otimes \cdots \otimes a_i) \bigotimes (a_{i+1} \otimes \cdots \otimes a_k) \quad (1)$$

Consider the Lie algebra $CoDer(C(A[1]))$ of all coderivations of this coalgebra. As the coalgebra is free, any coderivation D (if it is graded) is uniquely defined by a map $\Psi_D: A^{\otimes k} \rightarrow A$, and the degree of this coderivation is $k - 1$ (in conditions that A is not graded). The bracket $[\Psi_{D_1}, \Psi_{D_2}]$ is again a coderivation. Define the Hochschild Lie algebra as $\text{Hoch}^\bullet(A) = CoDer^\bullet(C(A[1]))$. To define the complex structure on it, consider

the particular coderivation D_m of degree $+1$ from the product $m: A^{\otimes 2} \rightarrow A$, which is the product in the associative algebra A . The condition $[D_m, D_m] = 0$ is equivalent to the associativity of m . Define the differential on $CoDer^\bullet(C(A[1]))$ as $d(\Psi) = [D_m, \Psi]$. In this way we get a dg Lie algebra. The differential is called the Hochschild differential, and the bracket is called the Gerstenhaber bracket. The definition of these structures given here is due to J.Stasheff.

1.2 The explicit definition

Here we relate the Stasheff's definition of the Hochschild cohomological complex with the usual one.

The concept of a coderivation of a (co)free coalgebra is dual to the concept of a derivation of a free algebra. Let L be a vector space, and let $T(L)$ be the free tensor algebra generated by the vector space L . Let $D: T(L) \rightarrow T(L)$ be a derivation, then it is uniquely defined by its value $D_L: L \rightarrow T(L)$ on the generators, and any D_L defines a derivation D of the free algebra $T(L)$. If we would like to consider only graded derivations, we restrict ourselves by the maps $D_L: L \rightarrow L^{\otimes k}$ for $k \geq 0$.

Dually, a coderivation D of the cofree coalgebra $C(P)$ cogenerated by a vector space P is uniquely defined by the restriction to cogenerators, that is, by a map $D_P: C(P) \rightarrow P$, or, if we consider the graded coderivations, the map D_P is a map $D_P: P^{\otimes k} \rightarrow P$ for $k \geq 0$.

In our case of the definition of the cohomological Hochschild complex of an associative algebra A , we have $P = A[1]$. Then the coderivations of the grading k form the vector space $Hoch^k(A) = \text{Hom}(A^{\otimes(k+1)}, A)$, $k \geq -1$. Now we can deduce the differential and the Gerstenhaber bracket from the Stasheff's construction. The answer is the following:

For $\Psi \in \text{Hom}(A^{\otimes k}, A)$ the cochain $d\Psi \in \text{Hom}(A^{\otimes(k+1)}, A)$ is given by the formula:

$$\begin{aligned} d\Psi(a_0 \otimes \cdots \otimes a_k) &= a_0 \Psi(a_1 \otimes \cdots \otimes a_k) + \\ &+ \sum_{i=0}^{k-1} (-1)^{i+1} \Psi(a_0 \otimes \cdots \otimes a_{i-1} \otimes (a_i a_{i+1}) \otimes a_{i+2} \otimes \cdots \otimes a_k) + \\ &+ (-1)^{k+1} \Psi(a_0 \otimes \cdots \otimes a_{k-1}) a_k \end{aligned} \quad (2)$$

For $\Psi_1 \in \text{Hom}(A^{\otimes(k+1)}, A)$ and $\Psi_2 \in \text{Hom}(A^{\otimes(l+1)}, A)$ the bracket $[\Psi_1, \Psi_2] = \Psi_1 \circ \Psi_2 - (-1)^{kl} \Psi_2 \circ \Psi_1$ where

$$\begin{aligned} (\Psi_1 \circ \Psi_2)(a_0 \otimes \cdots \otimes a_{k+l}) &= \\ \sum_{i=0}^k (-1)^{il} \Psi_1(a_0 \otimes \cdots \otimes a_{i-1} \otimes \Psi_2(a_i \otimes \cdots \otimes a_{i+l}) \otimes a_{i+l+1} \otimes \cdots \otimes a_{k+l}) \end{aligned} \quad (3)$$

1.3 The (co)bar-complex

Here we recall the definition of the (co)bar-complex of an associative (co)algebra. When the (co)algebra contains (co)unit, the (co)bar-complex is acyclic, and when the (co)algebra is the kernel of the augmentation, this concept is closely related to the Koszul duality.

Let A be an associative algebra. Then its bar-complex is

$$\cdots \rightarrow A^{\otimes 3} \rightarrow A^{\otimes 2} \rightarrow A \rightarrow 0$$

where $\deg A^{\otimes k} = -k + 1$, and the differential $d: A^{\otimes k} \rightarrow A^{\otimes(k-1)}$ is given as follows:

$$d(a_1 \otimes \cdots \otimes a_k) = (a_1 a_2) \otimes a_3 \otimes \cdots \otimes a_k - a_1 \otimes (a_2 a_3) \otimes \cdots \otimes a_k + \cdots + (-1)^k a_1 \otimes \cdots \otimes a_{k-2} \otimes (a_{k-1} a_k) \quad (4)$$

If the algebra A has unit, the bar-complex of A is acyclic in all degrees. Indeed, the map

$$a_1 \otimes \cdots \otimes a_k \mapsto 1 \otimes a_1 \otimes \cdots \otimes a_k$$

is a contracting homotopy.

Suppose now that the algebra A does not contain unit, and $A = B^+$ is the kernel of an augmentation map $\varepsilon: B \rightarrow \mathbb{C}$. (The map ε is a surjective map of algebras, in particular, it maps 1 to 1). Then the cohomology of the bar-complex of A is equal to the dual space $\text{Ext}_{B\text{-}Mod}^{\bullet}(\mathbb{C}, \mathbb{C})$.

Indeed, for any B -module M , we have the following free resolution of M :

$$\cdots B \otimes \overline{B} \otimes \overline{B} \otimes M \rightarrow B \otimes \overline{B} \otimes M \rightarrow B \otimes M \rightarrow M \rightarrow 0 \quad (5)$$

with the differential analogous to the bar-differential.

Consider the case $M = \mathbb{C}$. We can compute $\text{Ext}_{B\text{-}Mod}^{\bullet}(\mathbb{C}, \mathbb{C})$ using this resolution. In the answer we get the cohomology of the complex dual to the bar-complex of $A = B^+$.

The complex dual to the bar-complex of A is the *cobar-complex* for the coalgebra A^* . This cobar-complex is an associative dg algebra, and it is a free algebra, which by previous is a free resolution of the algebra $\text{Ext}_{B\text{-}Mod}(\mathbb{C}, \mathbb{C})$. For the sequel we write down explicitly the cobar-complex of a coassociative coalgebra Q :

$$0 \rightarrow Q \rightarrow Q \otimes Q \rightarrow Q \otimes Q \otimes Q \rightarrow \cdots \quad (6)$$

and the differential $\delta Q^{\otimes k} \rightarrow Q^{\otimes(k+1)}$ is

$$\delta(q_1 \otimes \cdots \otimes q_k) = (\Delta q_1) \otimes q_2 \otimes \cdots \otimes q_k - q_1 \otimes (\Delta q_2) \otimes \cdots \otimes q_k + \cdots + (-1)^{k-1} q_1 \otimes \cdots \otimes q_{k-1} \otimes (\Delta q_k) \quad (7)$$

where $\Delta: Q \rightarrow Q^{\otimes 2}$ is the coproduct.

In the case when $B = S(V)$ is the symmetric algebra, $\text{Ext}_{B\text{-}Mod}(\mathbb{C}, \mathbb{C})$ is the exterior algebra $\Lambda(V^*) = S(V^*[-1])$, and vice versa. In this way, we get a free resolution of the symmetric (exterior) algebra.

Example. Here we construct explicitly the free cobar-resolution \mathcal{R}^\bullet of the algebra $\mathbb{C}[x_1, x_2]$ of polynomials on two variables. As a graded algebra, \mathcal{R}^\bullet is the free algebra $\mathcal{R}^\bullet = \text{Free}(x_1, x_2, \xi_{12})$ where $\deg x_1 = \deg x_2 = 0$, $\deg \xi_{12} = -1$. The differential is 0 on x_1, x_2 , $d(\xi_{12}) = x_1 \otimes x_2 - x_2 \otimes x_1$, and satisfies the graded Leibniz rule. In degree 0 we have the tensor algebra $T(x_1, x_2)$, differential is 0 on degree 0 (there are no elements in degree 1). In degree -1, a general element is a non-commutative word in x_1, x_2, ξ_{12} in which ξ_{12} occurs exactly one time. For example, it could be a word $x_2 \otimes x_1 \otimes x_2 \otimes \xi_{12} \otimes x_1 \otimes x_2$. The image of the differential is then exactly the two-sided ideal in the tensor algebra $T(x_1, x_2)$ generated by $x_1 \otimes x_2 - x_2 \otimes x_1$. Then, the 0-th cohomology is $\mathbb{C}[x_1, x_2]$. It follows from the discussion above that all higher cohomology is 0.

1.4 The main construction

Here we construct a quasi-isomorphic map of dg Lie algebras $\Phi: \text{Hoch}^\bullet(S(V))/\mathbb{C} \rightarrow \text{Der}(\text{CoBar}^\bullet(S(V^*)^+))/\text{Inn}(\text{CoBar}^\bullet(S(V^*)^+))$.

Let $\Psi \in \text{Hom}((S(V))^{\otimes k}, S(V))$ be a k -cochain. Denote $V^* = W$, then we can consider Ψ the corresponding cochain in $\text{Hom}(S(W), (S(W))^{\otimes k})$. Here we consider $S(W)$ as *coalgebra*. Then this cochain may be considered as a derivation in $\text{Der}(\text{CoBar}^\bullet(S(W)))$. We would like to attach to it a derivation in $\text{Der}(\text{CoBar}^\bullet(S(W)^+))$, maybe modulo an inner derivation. So, we would like to show that there exist a map $\Phi: \text{Der}(\text{CoBar}^\bullet(S(W))) \rightarrow \text{Der}(\text{CoBar}^\bullet(S(W)^+))$ such that the diagram

$$\begin{array}{ccc} \text{Der}(\text{CoBar}^\bullet(S(W))) & \xrightarrow{\delta} & \text{Der}(\text{CoBar}^\bullet(S(W))) \\ \downarrow \Phi & & \downarrow \Phi \\ \text{Der}(\text{CoBar}^\bullet(S(W)^+)) & \xrightarrow{\delta} & \text{Der}(\text{CoBar}^\bullet(S(W)^+)) \end{array} \quad (8)$$

is commutative *modulo inner derivations* (here δ is the cobar differential).

In the coalgebra $S(W)$ the coproduct is given by the formula

$$\Delta(x_1 \dots x_k) = 1 \otimes (x_1 \dots x_k) + \sum_{\{i_1 \dots i_a\} \sqcup \{j_1 \dots j_b\} = \{1 \dots k\}} (x_{i_1} \dots x_{i_a}) \otimes (x_{j_1} \dots x_{j_b}) + (x_1 \dots x_k) \otimes 1 \quad (9)$$

and in the coalgebra $S(W)^+$ the coproduct is given by the same formula without the first and the last summands, which contain 1's.

Therefore the projection $p: S(W) \rightarrow S(W)^+$ is a map of coalgebras (dual to the imbedding of algebras), and the imbedding $i: S(W)^+ \rightarrow S(W)$ is *not*.

If $\Psi: S(W) \rightarrow S(W)^{\otimes k}$ is as above, we define $(\Phi(\Psi))(\sigma) = p^{\otimes k}(\Psi(i(\sigma))) \in \text{Hom}(S(W)^+, (S(W)^+)^{\otimes k})$.

Now we check the commutativity of the diagram (8) modulo inner derivations. It is clear that

$$(\Phi \circ \delta)(\sigma) - (\delta \circ \Phi)(\sigma) = p^{\otimes k}(\Psi(1)) \otimes \sigma \pm \sigma \otimes p^{\otimes k}(\Psi(1)) \quad (10)$$

which is an inner derivation $ad(p^{\otimes k}(\Psi(1)))$.

We have defined a map $\Phi_1: \text{Der}(CoBar^\bullet(S(W))) \rightarrow \text{Der}(CoBar^\bullet(S(W)^+)/\text{Inn}(CoBar^\bullet(S(W)^+)))$. The first dg Lie algebra is clearly isomorphic to the Hochschild cohomological complex of the algebra $S(V)$ modulo constants, and we can consider the map Φ_1 as a map

$$\Phi_1: \text{Hoch}^\bullet(S(V))/\mathbb{C} \rightarrow \text{Der}(CoBar^\bullet(S(V^*)^+)/\text{Inn}(CoBar^\bullet(S(V^*)^+)))$$

Let us note that the cobar complex $CoBar^\bullet(S(V^*)^+)$ is a free resolution of the Koszul dual algebra $\Lambda(V^*)$.

Now we have the following result:

Proposition. *The map Φ_1 is a quasi-isomorphism of dg Lie algebras.*

It is clear that Φ_1 is a map of dg Lie algebras, one only needs to proof that it is a quasi-isomorphism of complexes. Although it is useful to have this Proposition in mind when reading Section 2, we will not use it. The proof will appear somewhere.

2 Applications to deformation quantization

2.1 A lemma

We start with the following lemma, which is a formal version of the semi-continuity of cohomology of a complex depending on a parameter, which says that in a "singular value" of the parameter the cohomology may only raise:

Lemma. *Let \mathcal{R}^\bullet be a $\mathbb{Z}_{\leq 0}$ -graded complex with differential d , such that $H^i(\mathcal{R}^\bullet)$ vanishes for all $i \neq 0$. Consider $\mathcal{R}_\hbar^\bullet = \mathcal{R}^\bullet \otimes \mathbb{C}[[\hbar]]$. Let $d_\hbar: \mathcal{R}_\hbar^\bullet \rightarrow \hbar \mathcal{R}_\hbar^{\bullet+1}$ be a linear map of degree +1 such that*

$$(d + d_\hbar)^2 = 0$$

Then the cohomology H_\hbar^i of the complex $\mathcal{R}_\hbar^\bullet$ with the differential $d + d_\hbar$ vanishes for $i \neq 0$, and as a vector space, $H_\hbar^0(\mathcal{R}_\hbar^\bullet, d + d_\hbar) \simeq H^0(\mathcal{R}^\bullet, d) \otimes \mathbb{C}[[\hbar]]$.

Proof. Consider the filtration

$$\mathcal{R}_\hbar^\bullet \supset \hbar \mathcal{R}_\hbar^\bullet \supset \hbar^2 \mathcal{R}_\hbar^\bullet \supset \dots$$

of the complex $\mathcal{R}_\hbar^\bullet$ with the differential $d + d_\hbar$. Compute the cohomology of $\mathcal{R}_\hbar^\bullet$ by the spectral sequence corresponding to this filtration. The term $E_0^{p,q} = \hbar^p \mathcal{R}_\hbar^{p+q} / \hbar^{p+1} \mathcal{R}_\hbar^{p+q}$, and d_\hbar acts by 0 on $E_0^{p,q}$. Therefore, the cohomology in this term is the cohomology of the differential d and is $E_1^{p,-p} = \hbar^p H^0(\mathcal{R}^\bullet, d) / \hbar^{p+1} H^0(\mathcal{R}^\bullet, d)$ and $E_1^{p,q} = 0$ for $q \neq -p$. All higher differentials are 0 by the dimensional reasons, and the spectral sequence collapses in the term E_1 . The spectral sequence clearly converges to the cohomology of $\mathcal{R}_\hbar^\bullet$.

Lemma is proven. \square

2.2 A proof of the classical Poincaré-Birkhoff-Witt theorem

Let \mathfrak{g} be a Lie algebra. Its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is defined as the quotient-algebra of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal generated by elements $a \otimes b - b \otimes a - [a, b]$ for any $a, b \in \mathfrak{g}$. The Poincaré-Birkhoff-Witt theorem says that $\mathcal{U}(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g})$ as a \mathfrak{g} -module. We suggest here a (probably new) proof of this classical theorem, which certainly is not the simplest one, but sheds some light on the cohomological nature of the theorem.

Before starting with the proof, let us make some remark. Let us generalize the universal enveloping algebra as follows. Consider the tensor algebra $T(x_1, \dots, x_n)$ and its quotient $A_{c_{ij}^k}$ by the two-sided ideal generated by the relations $x_i \otimes x_j - x_j \otimes x_i - \sum_k c_{ij}^k x_k$, $1 \leq i < j \leq n$, where c_{ij}^k are not supposed to satisfy the Jacobi identity

$$\sum_a (c_{ij}^a c_{ak}^b + c_{jk}^a c_{ai}^b + c_{ki}^a c_{aj}^b) = 0 \quad (11)$$

Then, if (11) is not satisfied, the algebra $A_{c_{ij}^k}$ is smaller than $S(x_1, \dots, x_k)$, that is, the two-sided ideal, generated by the relations, is bigger than in the Lie algebra case when (11) is satisfied.

Now we pass to the proof. Let \mathfrak{g} be a Lie algebra. By the discussion in Section 1.3, $CoBar^\bullet(\Lambda^+(\mathfrak{g}))$ is a free resolution of the symmetric algebra $S(\mathfrak{g})$. Denote the cobar-differential by d . Introduce in $CoBar_\hbar^\bullet = CoBar^\bullet(\Lambda^+(\mathfrak{g})) \otimes \mathbb{C}[[\hbar]]$ a new differential $d + d_\hbar$, where $d_\hbar: CoBar_\hbar^\bullet \rightarrow \hbar \cdot CoBar_\hbar^{\bullet+1}$ comes from the chain differential in the Lie homology complex $\partial: \Lambda^i(\mathfrak{g}) \rightarrow \Lambda^{i-1}(\mathfrak{g})$. We denote

$$d_\hbar = \hbar \partial$$

The equation $(d + d_\hbar)^2 = 0$ follows from the fact that the chain Lie algebra complex is a dg coalgebra, and, therefore, its cobar-complex is well-defined.

Now, by Lemma 2.1, the complex $CoBar_\hbar^\bullet$ has only 0 degree cohomology, which is isomorphic to $H^0(CoBar^\bullet(\Lambda^+(\mathfrak{g}))) \otimes \mathbb{C}[[\hbar]] = S(\mathfrak{g}) \otimes \mathbb{C}[[\hbar]]$ as a (filtered) vector space. On the other hand, we can compute 0-th cohomology of $(CoBar_\hbar^\bullet(\Lambda^+(\mathfrak{g})), d + d_\hbar)$ directly. It is the quotient of the tensor algebra $T(\mathfrak{g}) \otimes \mathbb{C}[[\hbar]]$ by the two-sided ideal generated by the relations $a \otimes b - b \otimes a - \hbar[a, b]$, $a, b \in \mathfrak{g}$.

The specialization of the last isomorphism for $\hbar = 1$ gives the Poincaré-Birkhoff-Witt theorem.

2.3

Consider the following sequence of maps:

$$\begin{aligned} T_{poly}(V^*) &\xrightarrow{\mathcal{U}_S} \text{Hoch}(S(V)) \simeq \text{Der}(CoBar(S(V^*))) \xrightarrow{\Phi_1} \\ &\xrightarrow{\Phi_1} \text{Der}(CoBar(S^+(V^*))) / \text{Inn}(CoBar(S^+(V^*))) \end{aligned} \quad (12)$$

Here the first map is the Kontsevich formality L_∞ morphism for the algebra $S(V)$, the second isomorphism follows from the Stasheff's construction, and the third map is the map Φ_1 defined in Section 1.4.

Apply now the composition (12) to the vector space $V[1]$ instead of V^* .

Lemma. *Let V be a finite-dimensional vector space. Then there is a canonical isomorphism of the graded Lie algebras $T_{poly}(V^*) \simeq T_{poly}(V[1])$.*

Proof. It is straightforward. The map maps k -polyvector field with constant coefficients on V^* to a k -linear function on $V[1]$, and so on. \square

Remark. The algebras $S(V)$ and $\Lambda(V^*[-1])$ are Koszul dual, and they have isomorphic Hochschild comology with all structures (see [Kel]).

Denote by K the correspondence $K: T_{poly}(V^*) \rightarrow T_{poly}(V[1])$ from Lemma. Let α be a polynomial Poisson bivector on the space V^* . By the correspondence from Lemma, we get a polyvector field $K(\alpha)$ which in general is not a bivector, but still satisfies the Maurer-Cartan equation

$$[K(\alpha), K(\alpha)] = 0 \quad (13)$$

Let us rewrite (12) for $V[1]$:

$$\begin{aligned} T_{poly}(V[1]) &\xrightarrow{\mathcal{U}_\Lambda} \text{Hoch}(\Lambda(V^*)) \simeq \text{Der}(\text{CoBar}(\Lambda(V))) \xrightarrow{\Phi_1} \\ &\xrightarrow{\Phi_1} \text{Der}(\text{CoBar}(\Lambda^-(V)))/\text{Inn}(\text{CoBar}(\Lambda^-(V))) \end{aligned} \quad (14)$$

The composition (14) maps the polyvector $\hbar K(\alpha)$ to a derivation d_\hbar of degree $+1$ in $\text{Der}(\text{CoBar}(\Lambda^-(V))) \otimes \mathbb{C}[[\hbar]]$, which satisfies the Maurer-Cartan equation

$$(d + d_\hbar)^2 = 0 \quad (15)$$

in Der/Inn , where d is the cobar-differential.

Actually, (15) is satisfied in $\text{Der}(\text{CoBar}(\Lambda^-(V))) \otimes \mathbb{C}[[\hbar]]$, not only in $\text{Der}(\text{CoBar}(\Lambda^-(V))) \otimes \mathbb{C}[[\hbar]]/\text{Inn}(\text{CoBar}(\Lambda^-(V))) \otimes \mathbb{C}[[\hbar]]$. Indeed, we suppose that V is placed in degree 0, then $\text{CoBar}(\Lambda^-(V))$ is $\mathbb{Z}_{\leq 0}$ -graded. Therefore, any inner derivation has degree ≤ 0 , while d_\hbar has degree $+1$. We have the following

2.4

Lemma. *Let α be a Poisson bivector on V^* , and let $K(\alpha)$ be the corresponding Maurer-Cartan polyvector of degree 1 in $T_{poly}(V[1])$. Then (14) defines an \hbar -linear derivation d_\hbar of $\text{CoBar}(\Lambda^-(V)) \otimes \mathbb{C}[[\hbar]]$ of degree $+1$ corresponding to $\hbar K(\alpha)$, such that*

$$(d + d_\hbar)^2 = 0$$

where d is the cobar-differential. Moreover, d_{\hbar} obeys

$$d_{\hbar}(\xi_i \wedge \xi_j) = \hbar \text{Sym}(\alpha_{ij}) + \mathcal{O}(\hbar^2) \quad (16)$$

where $\xi_i \wedge \xi_j \in \Lambda^2(V)$, $\text{Sym}(\alpha_{ij}) \in T(V)$ is the symmetrization, and $\{\xi_i\}$ is the basis in $V[1]$ dual to the basis $\{x_i\}$ in V^* in which $\alpha = \sum_{ij} \alpha_{ij} \partial_i \wedge \partial_j$

Proof. We only need to prove (16), all other statements are already proven. We prove it in details in Section 2.7. \square

2.5

Let A be an \hbar -linear associative algebra which is the quotient of the tensor algebra $T(V) \otimes \mathbb{C}[[\hbar]]$ of a vector space V by the two-sided ideal generated by relations

$$x \otimes y - y \otimes x = R(x, y)$$

for any $x, y \in V$, where $R(x, y) \in \hbar T(V) \otimes \mathbb{C}[[\hbar]]$. Consider the following filtration:

$$A \supset \hbar A \supset \hbar^2 A \supset \hbar^3 A \supset \dots \quad (17)$$

This is clearly an algebra filtration: $(\hbar^k A) \cdot (\hbar^\ell A) \subset \hbar^{k+\ell} A$. Consider the associated graded algebra $\text{gr} A$. We say that the algebra A is a Poincaré-Birkhoff-Witt (PBW) algebra if $\text{gr} A \simeq S(V) \otimes \mathbb{C}[[\hbar]]$ as a graded $\mathbb{C}[[\hbar]]$ -linear algebra.

In general, $\text{gr} A$ is less than $S(V) \otimes \mathbb{C}[[\hbar]]$, it is a quotient of $S(V) \otimes \mathbb{C}[[\hbar]]$. One can say that the PBW property is equivalent to the property that the quotient-algebra has "the maximal possible size".

2.6

Let α be a polynomial Poisson bivector in V^* . In Sections 2.3 and 2.4 we constructed an \hbar -linear derivation d_{\hbar} on $\text{CoBar}(\Lambda^-(V)) \otimes \mathbb{C}[[\hbar]]$ such that $(d + d_{\hbar})^2 = 0$ where d is the cobar-differential. By Section 1.3, the cobar-complex $\text{CoBar}(\Lambda^-(V))$ is a free resolution of the algebra $S(V)$, in particular, the cohomology of d does not vanish only in degree 0 where it is equal to $S(V)$. We are in the situation of Lemma 2.1. In particular, the dg algebra $(\text{CoBar}(\Lambda^-(V)) \otimes \mathbb{C}[[\hbar]], d + d_{\hbar})$ has only non-vanishing cohomology in degree 0, and this 0-degree cohomology is an algebra, which is a PBW algebra by Lemma 2.1.

Theorem. *The construction above constructs from a Poisson polynomial bivector α on V^* an algebra A_{α} with generators x_1, \dots, x_n and relations $[x_i, x_j] = d_{\hbar}(\xi_i \wedge \xi_j)$. This algebra is a PBW algebra.* \square

Conjecture. *The algebra A_{α} is isomorphic to the Kontsevich star-algebra on $S(V) \otimes \mathbb{C}[[\hbar]]$ constructed from the Poisson bivector α . (We suppose that in the formality morphisms $\mathcal{U}_{\Lambda}: T_{\text{poly}}(V[1]) \rightarrow \text{Hoch}(\Lambda(V^*))$ in (14), and $\mathcal{U}_S: T_{\text{poly}}(V^*) \rightarrow \text{Hoch}(S(V))$ which is used in the construction of the star-product, one uses the same propagator in the definition of the Kontsevich integrals, see [K97]).*

In our approach, we lift this Conjecture on the level of complexes, and get a diagram commutative in the homotopical category $\mathcal{H}om_{dg}$ from [Sh]. The commutativity of this diagram implies the Conjecture.

2.7 An explicit formula

One can write down explicitly the relations in the algebra A_α , in the terms of the Kontsevich integrals [K97]. For this we need to find explicit formula for the \hbar -linear derivation d_\hbar in $CoBar(\Lambda^-(V)) \otimes \mathbb{C}[[\hbar]]$. Here we suppose some familiarity with [K97].

First of all, recall how the Kontsevich deformation quantization formula is written. Let α be a Poisson structure on V^* . Then the formula is

$$f \star g = f \cdot g + \sum_{k \geq 1} \hbar^k \left(\sum_{m \geq 1} \frac{1}{m!} \sum_{\Gamma \in G_{2,m}^2} W_\Gamma U_\Gamma(\alpha, \dots, \alpha) \right) \quad (18)$$

Here Γ is an admissible graph with two vertices on the "real line" and m vertices in the upper half-plane, and with 2 outgoing edges at each vertex in the upper half-plane, that is, $\Gamma \in G_{2,m}^2$, in particular, it is an oriented graph with $2m$ edges; W_Γ is the Kontsevich integral of the Graph Γ . *Let us note that the all graphs involved in (18) may have arbitrary many incoming edges at each vertex at the upper half-plane, and exactly two outgoing edges.*

Now let $\alpha = \sum_{ij} \alpha_{ij} \partial_i \wedge \partial_j$, where $\alpha_{ij} = \sum_I c_{ij}^I x_{i_1} \dots x_{i_k}$ (I is a multi-index).

Then the "Koszul dual" polyvector $K(\alpha)$ is a polyvector field with quadratic coefficients:

$$K(\alpha) = \sum_{i,j,I} c_{ij}^I (\xi_i \xi_j) \cdot \partial_{\xi_{i_1}} \wedge \dots \wedge \partial_{\xi_{i_k}} \quad (19)$$

It has total degree 1 and satisfies the Maurer-Cartan equation.

Firstly we write the formula for the image of $K(\hbar\alpha)$ by the Kontsevich formality, that is, denote by

$$\mathcal{U}(K(\alpha)) = \hbar U_1(K(\alpha)) + \hbar^2 \frac{1}{2} U_2(K(\alpha), K(\alpha)) + \dots + \hbar^k \frac{1}{k!} U_k(K(\alpha), \dots, K(\alpha)) + \dots \quad (20)$$

We can write down explicitly this formula in graphs. *Let us note that the graphs involving in (20) may have arbitrary many outgoing edges in the vertices at the upper half-plane, but exactly two, one, or 0 incoming edges, because all components of $K(\alpha)$ are quadratic polyvector fiels.* That is, in a sense the graphs in (20) are "dual" to the graphs in (18).

Let us note also that the right-hand side of (20) is a polydifferential operator in $\text{Hoch}(\Lambda(V^*))$ of non-homogeneous Hochschild degree, but of the total (Hochschild degree and Λ -degree) +1.

Now we should apply to $\mathcal{U}(K)$ our map Φ_1 to get a derivation of the cobar-complex $CoBar(\Lambda^-(V))$. After this, we get the final answer for d_\hbar .

Let us compute its component in the first power of \hbar . It is just the Hochschild-Kostant-Rosenberg map of $U_1(K(\alpha))$ which is the symmetrization map $\text{Sym}: S(V) \rightarrow T(V)$ in this case.

Let us note that in the case of a quadratic Poisson structure algebra the relations $R_{ij} \in (V \otimes V) \otimes \mathbb{C}[[\hbar]] \subset T(V) \otimes \mathbb{C}[[\hbar]]$ are quadratic.

One can imagine then the relations $x_i \otimes x_j - x_j \otimes x_i = d_\hbar(\xi_i \wedge \xi_j)$ hold *exactly* in the Kontsevich star-algebra defined from the same propagator. It would be a "pure duality", which would be very nice. Our Conjecture 2.6 is a weaker statement, that this relations hold in an algebra gauge equivalent to the Kontsevich star-algebra. Anyway, these relations may considered as some very non-trivial relations with Kontsevich integrals, which are transcendental numbers which in general is almost impossible to compute directly.

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References

- [GJ] P. Goerss, J. Jardine, Simplicial Homotopy Theory, a book, Birkhäuser, Progress in Math., Vol. 174, 1999,
- [H1] V. Hinich, Homological algebra of homotopy algebras, preprint math/9702015,
- [H2] V. Hinich, DG coalgebras as formal stacks, preprint math/9812034,
- [K97] M. Kontsevich, Deformation quantization of Poisson manifolds I, preprint q-alg/9709040,
- [K99] M. Kontsevich, Operads and Motives in deformation quantization, preprint q-alg/9904055,
- [Kel] B. Keller, Derived invariance of higher structures on the Hochschild complex, available at www.math.jussieu.fr/~keller,
- [Q] D. Quillen, Rational homotopy type, Annals of Mathematics, 2nd Ser., Vol.90, No.2 (1969), pp. 205-295,
- [Q1] D. Quillen, Homotopical Algebra, Springer Lecture Notes in Math., vol. 43, 1967,
- [PP] A. Polishchuk, L. Positselski, Quadratic Algebras, a book, AMS University Lecture Series, 2005,

- [Sh] B. Shoikhet, An explicit construction of the Quillen homotopical category of dg Lie algebras, preprint math.07061333,
- [T] D. Tamarkin, Action of the Grothendieck-Teichmüller group on the operad of Gerstenhaber algebras.

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